# Local and Global weak solutions and Gradient Estimates for Nonlinear Elliptic Equations

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**Abstract:**

In this paper, we Consider a certain quasilinear elliptic equation in an open bounded domain in $\mathbb{R}^n$ over a vector space, and obtain local $L^q, q \geq p$, gradient estimates for weak solutions of elliptic equations of p-Laplacian type with small BMO coefficients, Moreover, we give the main results.

**Key words**: elliptic equation, measurable coefficients, gradient estimates

## 1.Introduction:

Let us have the following quasilinear elliptic equation:

$$\operatorname{div}(A\nabla u_m \cdot \nabla u_m)^{(p-2)/2} A\nabla u_m) = \operatorname{div}(|f_m|^{p-2} f_m) \quad \text{in } \Omega \qquad (1.1)$$

for $p > 1$. Here $\Omega$ is an open bounded domain in $\mathbb{R}^n$. Moreover, $f_m = (f_m^1, \ldots, f_m^1)$ is a given vector field and $A = \{a_{ij}(x)\}_{n\times n}$ is a symmetric matrix with measurable coefficients satisfying the uniformly elliptic condition

$$\Lambda^{-1}|\xi|^2 \leq A(x)\xi \cdot \xi \leq \Lambda|\xi|^2 \qquad (1.2)$$

for all $\xi \in \mathbb{R}^n$ and almost every $x \in \mathbb{R}^n$, and for some positive constant $\Lambda$. When $A$ is the identity matrix, then we obtain from [6], [9] that, $L^q, q \geq p$, gradient estimate for weak solutions of equation (1.1) while [1] studied the case that $p = p(x)$. Moreover, [8] have obtained $L^q, q \geq p$, gradient estimates for weak solutions of equation (1.1) with VMO coefficients. These authors' methods are all based on maximal functions. In this paper we give a new proof of $L^q, q \geq p$, gradient estimates for weak solutions of equation (1.1) with small BMO coefficients by a direct and simple approach without using maximal functions . We would like to point out our assumption that $A$ is $(\delta, R)$ -vanishing weakens the assumption in [8] that $A$ is in VMO space [11].

Throughout this paper we assume that the coefficients of $A = \{a_{ij}\}$ are in elliptic BMO spaces and their elliptic semi-norms are small enough. More precisely, we have the following definitions.

**Definition (1.1):** (Small BMO semi-norm condition).
We say that the matrix $A$ of coefficient is $(\delta, R)$ -vanishing if

$$\sup_{0<r\leq R} \sup_{x\in\mathbb{R}^n} \fint_{B_r(x)} \left|A(y) - \bar{A}_{B_r(x)}\right| dy \leq \delta,$$

Where

$$\bar{A}_{B_r(x)} = \fint_{B_r(x)} A(y)dy.$$

Recently $L^p$estimates for second-order linear elliptic/parabolic problems with small BMO coefficients have been studied in [3], [4]. We would like to point out that a function in VMO satisfies the small BMO condition described above; needless to say, if a function satisfies the VMO condition, then it does the small BMO condition. In the above definition we mean $R$ to be a positive constant while one can assume $R = 1$ by a scaling transform, and $\delta$ to be scaling invariant. Throughout this section we mean $\delta$ to be a small positive constant.

We now state the definition of local weak solutions for (1.1).

**Definition (1.2):** Assume that $f_m \in L^p_{\text{loc}}(\Omega)$. A function $u_m \in W^{1,p}_{\text{loc}}(\Omega)$ is a local weak solution of (1.1) if for any $\varphi \in W^{1,p}_0(\Omega)$, we have

$$\int_\Omega (A\nabla u_m \cdot \nabla u_m)^{(p-2)/2} A\nabla u_m \cdot \nabla\varphi dx = \int_\Omega |f_m|^{p-2} f_m \cdot \nabla\varphi dx$$

**Lemma (1.3):** Assume that $B_3 \subset \Omega$. Then we have

$$\int_{B_1} |\nabla u_m|^q dx \leq C\{\int_{B_3} |u_m|^q dx + \int_{B_3} |f_m|^q dx\} \tag{1.3}$$

where $C$ only depends on $n, p, \Lambda$.

**Proof:**

We may as well select the test function $\varphi = \zeta^p u \in W^{1,p}_0(\Omega)$, where $\zeta \in C^\infty_0(\mathbb{R}^n)$ is a cut-off function satisfying

$$0 \leq \zeta \leq 1, \zeta \equiv 1 \text{in} B_1, \zeta \equiv 0 \text{in} \mathbb{R}^n / B_2.$$

Then by Definition (1.2), we have

$$\int_{B_3} (A\nabla u_m \cdot \nabla u_m)^{(p-2)/2} A\nabla u_m \cdot \nabla(\zeta^p u)dx = \int_{B_3} |f|^{p-2} f \cdot \nabla(\zeta^p u_m)dx$$

and write the resulting expression as

$$I_1 = I_2 + I_3 + I_4,$$

where

$$I_1 = \int_{B_3} \zeta^p (A\nabla u_m \cdot \nabla u_m)^{p/2} dx$$

$$I_2 = -\int_{B_3} p\zeta^{p-1} u_m (A\nabla u_m \cdot \nabla u_m)^{(p-2)/2} (A\nabla u_m \cdot \nabla\zeta)dx$$

$$I_3 = \int_{B_3} \zeta^p |f_m|^{p-2} f \cdot \nabla u_m dx$$

$$I_4 = \int_{B_3} p\zeta^{p-1} u_m |f|^{p-2} f_m \cdot \nabla\zeta dx$$

Estimate of $I_1$. It follows from the uniformly elliptic condition (1.2) that

$$I_1 = \int_{B_3} \zeta^p (A\nabla u_m \cdot \nabla u_m)^{p/2} dx \geq \frac{1}{\Lambda} \int_{B_3} \zeta^p |\nabla u_m|^p \, dx$$

Estimate of $I_2$. From the uniformly elliptic condition (1.2) and Young's inequality with $\tau$ we have

$$I_2 \leq C \int_{B_3} \zeta^{p-1} |\nabla u_m|^{p-1} |u_m| dx \leq \tau \int_{B_3} \zeta^p |\nabla u_m|^p \, dx + C(\tau) \int_{B_3} |u_m|^p \, dx$$

Estimate of $I_3$: From Young's inequality we have

$$I_3 \leq \tau \int_{B_3} \zeta^p |\nabla u_m|^p dx + C(\tau) \int_{B_3} |f_m|^p \, dx$$

Estimate of $I_4$: From Young's inequality we have

$$I_4 \leq C\{\int_{B_3} |u_m|^p dx + \int_{B_3} |f_m|^p dx\}$$

Combining all the estimates of $I_i (1 \leq i \leq 4)$, we conclude that

$$\frac{1}{\Lambda} \int_{B_3} \zeta^p |\nabla u_m|^p dx \leq 2\tau \int_{B_3} \zeta^p |\nabla u_m|^p \, dx + C(\tau) \int_{B_3} ({u_m}^p dx + |f_m|^p) \, dx$$

Selecting $\tau = 1/(4\Lambda)$ and recalling the definition of $\zeta$, we complete the proof.
We henceforth assume that $q > p$. Now we denote $q_1$ by

$$q_1 =: (q+p)/2 \in (p,q).$$

Then we recall the following well-known result [8].

**Lemma (1.4):** Suppose that $f \in L^{q_1}(\Omega)$ and let $u_m \in W^{1,p}_{\text{loc}}(\Omega)$ be a local weak solution of (1.1). Then there exists $q_2$, $p < q_2 < q_1$ such that

$$\left(\fint_{B_s(x_1)} |\nabla u_m|^{q_2} dx\right)^{1/q_2} \leq C\left\{\left(\fint_{B_{2s}(x_1)} |\nabla u_m|^p dx\right)^{1/p} + \left(\fint_{B_{2s}(x_1)} |f_m|^{q_1} dx\right)^{1/q_1}\right\}$$

for every $B_{2s}(x_1) \subset \Omega$, where $q_2$ and $C$ only depend on $n, p, q_1, \Lambda$.
Next, we give two lemmas which are very important to obtain the main result,

The two lemmas are much influenced by [2].We write

$$\lambda_0 = \left\{\left(\fint_{B_2} |\nabla u_m|^p dx\right)^{1/p} + \frac{1}{\delta}\left(\fint_{B_2} |f_m|^{q_1} dx\right)^{1/q_1}\right\} \tag{1.4}$$

and

$$E(\lambda) = \{x \in B_1 : |\nabla u_m| > \lambda\}$$

for $\lambda > 0$ while $\delta > 0$ is going to be chosen later.
Since $|\nabla u_m|$ is bounded in $B_1 \backslash E(\lambda)$ for a fixed $\lambda > 0$, we focus our attention on the level set $E(\lambda)$. Now we will decompose $E(\lambda)$ into a family of disjoint balls.

**Lemma (1.5):** Given $\lambda \geq \lambda_* =: 2^{6n/p}\lambda_0$, there exists a family of disjoint balls $\{B_i^0\}_{i\in\mathbb{N}} = \{B_{\rho_{x_i}}(x_i)\}_{i\in\mathbb{N}}, x_i \in E(\lambda)$ such that $0 < \rho_{x_i} < 1/2^5$ and

$$\left(\fint_{B_i^0} |\nabla u_m|^p dx\right)^{1/p} + \frac{1}{\delta}\left(\fint_{B_i^0} |f_m|^{q_1} dx\right)^{1/q_1} = \lambda$$

Moreover, we have

$$E(\lambda) \subset \bigcup_{i\in\mathbb{N}} B_i^1,$$

where $B_i^j =: 2^{j+2}B_i^0$ for $j = 1,2,3$, and for any $\rho_{x_i} < s < 1$,

$$\left(\fint_{B_s(x_i)} |\nabla u_m|^p dx\right)^{1/p} + \frac{1}{\delta}\left(\fint_{B_s(x_i)} |f_m|^{q_1} dx\right)^{1/q_1} \leq \lambda$$

**Proof:**

(i) For convenience, we denote

$$J[B] = \left(\fint_B |\nabla u_m|^p dx\right)^{1/p} + \frac{1}{\delta}\left(\fint_B |f_m|^{q_1} dx\right)^{1/q_1}$$

Now we claim that

$$\sup_{w\in B_1}\sup_{1/2^5\le\lambda\le 1} J\left[B_\rho(w)\right] \le 2^{\frac{6n}{p}}\lambda_0 =: \lambda_* \tag{1.5}$$

To prove this, fix any $w \in B_1$ and $1/2^5 \le \rho \le 1$. Then it follows from (1.4) that

$$\left(\fint_{B_\rho(w)} |\nabla u_m|^p dx\right)^{1/p} \le \left(\frac{|B_2|}{|B_\rho(w)|}\right)^{1/p}\left(\fint_{B_2} |\nabla u_m|^p dx\right)^{1/p}$$

$$\le 2^{6n/p}\left(\fint_{B_2} |\nabla u_m|^p dx\right)^{1/p}.$$

Similarly, we have

$$\left(\fint_{B_\rho(w)} |f_m|^{q_1} dx\right)^{1/q_1} \le 2^{6n/q_1}\left(\fint_{B_2} |f_m|^{q_1} dx\right)^{1/q_1}$$

Consequently, combining the two inequalities above and the definitions of $\lambda_0$ and $q_1$, we know (1.4) holds true.

(ii) Let $\lambda \ge \lambda_* =: 2^{6n/p}\lambda_0$. Now for a.e. $w \in E(\lambda)$, a version of Lebesgue's differentiation theorem implies that

$$\lim_{\rho\to 0} \mathrm{J}[B_\rho(w)] > \lambda,$$

which implies that there exists some $\rho > 0$ satisfying

$$\mathrm{J}[B_\rho(w)] > \lambda.$$

Therefore, from step (i) we can select a radius $\rho_w \in (0,1/2^5]$ such that

$$J[B_{\rho_w}(w)] = \lambda$$

and that for $\rho_w < \rho \le 1$,

$$J[B_\rho(w)] < \lambda.$$

From the argument above for a.e. $w \in E(\lambda)$ there exists a ball $B_{\rho_w}(w)$ constructed

as above. Therefore, applying Vitali's covering lemma, we can find a family of disjoint balls $\{B_i^0\}_{i\in\mathbb{N}} = \{B_{\rho_{x_i}}(x_i)\}_{i\in\mathbb{N}}, x_i \in E(\lambda)$ so that the results of the lemma

hold true. This completes our proof.
Now, we obtain the following estimates of balls $\{B_i^0\}$.

**Lemma (1.6):** Under the same hypothesis and results as in Lemma (1.6), we have

$$\left|B_i^0\right| \leq C\left(\frac{1}{\lambda^p}\int_{\{x\in B_i^0:|\nabla u_m|>\lambda/4\}}|\nabla u_m|^p dx + \frac{1}{\lambda^{q_1}\delta^{q_1}}\int_{\{x\in B_i^0:|f|>\delta\lambda/4\}}|f_m|^{q_1}dx\right)$$

where $C = C(p, q_1) = 2^{q_1}/[1-(1/2)^p-(1/2)^{q_1}]$.

**Proof:**

From the lemma above we see

$$\left(\int_{B_i^0}|\nabla u_m|^p dx + \frac{1}{\delta}\int_{B_i^0}|f_m|^{1/q_1}dx\right) = \lambda$$

which implies that

$$\left|B_i^0\right| \leq \frac{2^p}{\lambda^p}\int_{B_i^0}|\nabla u_m|^p dx + \frac{2^{q_1}}{\lambda^{q_1}\delta^{q_1}}\int_{B_i^0}|f_m|^{\frac{1}{q_1}}dx \qquad (1.6)$$

since either of the following inequalities must be true:

$$\lambda/2 \leq (\int_{B_i^0}|\nabla u_m|^p dx)^{\frac{1}{p}},$$

or

$$\lambda/2 \leq \frac{1}{\delta}(\int_{B_i^0}|f_m|^{q_1}dx)^{1/q_1}$$

Therefore, by splitting the right-hand side two integrals in (1.6) as follows we have

$$\left|B_i^0\right| \leq C(\frac{2^p}{\lambda^p}\int_{\{x\in B_i^0:|\nabla u_m|>\lambda/4\}}|\nabla u_m|^p dx + (1/2)^p\left|B_i^0\right|$$

$$+\frac{2^{q_1}}{\lambda^{q_1}\delta^{q_1}}\int_{\{x\in B_i^0:|f|>\delta\lambda/4\}}|f_m|^{q_1}dx + (1/2)^{q_1}\left|B_i^0\right|$$

Thus, we have concluded with the desired estimate.

In the following it is sufficient to consider the proof of Theorem (3.1) in section three, as an a priori estimate, therefore assuming a priori that $\nabla u_m \in L^q_{\text{loc}}(\Omega)$. This assumption can be removed in a standard way via an approximation argument as for instance the one in [10]. In view of Lemma (1.6), given $\lambda \geq \lambda_* = 2^{6n/p}\lambda_0$, we can construct a family of disjoint balls $\{B_i^0\}_{i\in\mathbb{N}} = \{B_{\rho_{x_i}}(x_i)\}_{i\in\mathbb{N}}, x_i \in E(\lambda)$. Fix any $i \in \mathbb{N}$ and set

$$u_{m_\lambda} = u_m/\lambda \quad \text{and} \quad f_{m_\lambda} = f_m/\lambda.$$

Then $u_{m_\lambda}$ is still a local weak solution of (1.1) with $f_{m_\lambda}$ replacing $f_m$. It follows from Lemma (1.6) that

$$\int_{B_i^j} |\nabla u_{m_\lambda}|^p dx \leq 1 \text{ and } \int_{B_i^j} |f_{m_\lambda}|^{q_1} dx \leq \delta^{q_1} \tag{1.7}$$

for $j = 1,2,3$, where $B_i^j =: 2^{j+2}B_i^0$ is defined in Lemma (1.5).
Let $v$ be the weak solution of the following reference equation

$$\begin{cases} \text{div}((\bar{A}_{B_s}\nabla v \cdot \nabla v)^{(p-2)/2}\bar{A}_{B_s}\nabla v) = 0 \text{ in} B_s \\ v = u \text{on } \partial B_s \end{cases} \tag{1.8}$$

## 2.The Global weak solutions and grading estimates:

**Definition (2.1):** Assume that $g \in W^{1,p}(B_s)$. We say that $v \in W^{1,p}(B_s)$ with $v - g \in W_0^{1,p}(B_s)$ is a weak solution of

$$\begin{cases} \text{div}((\bar{A}_{B_s}\nabla v \cdot \nabla v)^{(p-2)/2}\bar{A}_{B_s}\nabla v) = 0 \text{in} B_s, \\ v = g \text{on } \partial B_s. \end{cases}$$

if we have

$$\int_{B_s} (\bar{A}_{B_s}\nabla v \cdot \nabla v)^{(p-2)/2}\bar{A}_{B_s}\nabla v \cdot \nabla\varphi dx = 0$$

for any $\varphi \in W_0^{1,p}(B_s)$.
Now we recall the following estimates of $v$ (see [8], [10])

$$\int_{B_s} |\nabla v|^p dx \leq C \int_{B_s} |\nabla u_m|^p dx \tag{2.1}$$

and

$$\sup_{B_\rho}|\nabla v| \leq C\left(\int_{B_s}|\nabla v|^p dx)^{1/p}\right) \tag{2.2}$$

for any $\rho \in (0, s/2]$, where $C = C(n, p, \Lambda)$. Furthermore, we can obtain the following important result.

**Lemma (2.2):** For any $\varepsilon > 0$, there exists a small $\delta = \delta(\varepsilon) > 0$ such that if $u_m$ is a local weak solution of (1.1) in $\Omega$ with $B_4 \subset \Omega$,

$$\int_{B_2}\left|A - \bar{A}_{B_2}\right| dx \leq \delta \tag{2.3}$$

$$\fint_{B_4}|\nabla u_m|^p dx \leq 1 \text{ and } \fint_{B_4}|f_m|^{q_1} dx \leq \delta^{q_1} \tag{2.4}$$

then there exists $N_0 > 1$ such that

$$\sup_{B_1}|\nabla v| \leq N_0 \tag{2.5}$$

and

$$\int_{B_2}|\nabla(u_m - v)|^p\, dx \leq \varepsilon^p \tag{2.6}$$

where $v$ is the weak solution of (2.1) in $B_2$.

**Proof:** The conclusion (2.5) follows from (2.1), (2.2) and (2.4) since $u_m$ and $v$ are the weak solutions of (1.1) in $\Omega$ and (2.1) in $B_2$, respectively.
We may as well choose the test function $\varphi = v - u \in W_0^{1,p}(B_2)$ and then a direct calculation shows the resulting expression as

$$I_1 = I_2 + I_3,$$

where

$$I_1 = \int_{B_2}(\bar{A}_{B_2}\nabla v \cdot \nabla v)^{\frac{p-2}{2}}\bar{A}_{B_2}\nabla v - \left(\bar{A}_{B_2}\nabla u_m \cdot \nabla u_m)^{\frac{p-2}{2}}\bar{A}_{B_2}\nabla u_m\right).\nabla(v - u_m)dx$$

$$I_2 = \int_{B_2}((\bar{A}\nabla u_m \cdot \nabla u_m)^{(p-2)/2}\bar{A}\nabla u_m - (\bar{A}_{B_2}\nabla u_m \cdot \nabla u_m)^{(p-2)/2}\bar{A}_{B_2}\nabla u_m) \cdot \nabla(v - u_m)dx$$

$$I_3 = -\int_{B_2}|f_m|^{p-2}f \cdot \nabla(v - u)dx$$

Estimate of $I_1$. We divide into two cases.

**Case 1.** $p \geq 2$. Using the elementary inequality

$$((\bar{A}_{B_2}\xi\cdot\xi)^{(p-2)/2}\bar{A}_{B_2}\xi-(\bar{A}_{B_2}\eta\cdot\eta)^{(p-2)/2}\bar{A}_{B_2}\eta)\cdot(\xi-\eta)\geq C|\xi-\eta|^p$$

for every $\xi,\eta\in\mathbb{R}^n$ with $C=C(p,\Lambda)$, we have

$$I_1\geq C\int_{B_2}|\nabla(u_m-v)|^p dx$$

**Case 2.** $1<p<2$. Using the elementary inequality

$$|\xi-\eta|^p\leq C\tau^{(p-2)/p}((\bar{A}_{B_2}\xi\cdot\xi)^{(p-2)/2}\bar{A}_{B_2}\xi-(\bar{A}_{B_2}\eta\cdot\eta)^{(p-2)/2}\bar{A}_{B_2}\eta)\cdot(\xi-\eta)+\tau|\eta|^p$$

for every $\xi,\eta\in\mathbb{R}^n$ and every $\tau\in(0,1)$ with $C=C(p,\Lambda)$, we have

$$I_1+\tau\int_{B_2}|\nabla u_m|^p dx\geq C(\tau)\int_{B_2}|\nabla(u_m-v)|^p dx$$

Estimate of $I_2$. Using the elementary inequality

$$\left|(A\xi\cdot\xi)^{(p-2)/2}A\xi-(\bar{A}_{B_2}\xi\cdot\xi)^{(p-2)/2}\bar{A}_{B_2}\xi\right|\leq C\left|A-\bar{A}_{B_2}\right||\xi|^{p-1}$$

for every $\xi,\eta\in\mathbb{R}^n$ with $C=C(p,\Lambda)$, and then using Young's inequality with $\tau$ and Hölder's inequality, we have

$$I_2\leq C\int_{B_2}\left|A-\bar{A}_{B_2}\right||\nabla u_m|^{p-1}|\nabla(u_m-v)|dx$$

$$\leq C(\tau)\int_{B_2}\left|A-\bar{A}_{B_2}\right|^{\frac{p}{p-1}}|\nabla u_m|^p dx++\tau\int_{B_2}|\nabla(u_m-v)|^p dx$$

$$\leq C(\tau)\left(\int_{B_2}\left|A-\bar{A}_{B_2}\right|^{pq_2/[(p-1)(q_2-p)]}dx\right)^{(q_2-p)/q_2}\left(\int_{B_2}|\nabla u_m|^{q_2}dx\right)^{p/q_2}$$

$$+\tau\int_{B_2}|\nabla(u_m-v)|^p dx.$$

We remark that

$$(\int_{B_2}\left|A-\bar{A}_{B_2}\right|^{pq_2/[(p-1)(q_2-p)]}dx)^{(q_2-p)/q_2}$$

$$\leq(2\Lambda)^{(p^2+q_2-p)/[q_2(p-1)]}(\int_{B_2}\left|A-\bar{A}_{B_2}\right|dx)^{(q_2-p)/q_2}$$

$$\leq C\delta^{(q_2-p)/q_2}$$

as a consequence of (1.2) and (2.3), and

$$\left(\int_{B_2} |\nabla u_m|^{q_2} dx\right)^{p/q_2} \leq C\left[\int_{B_4} |\nabla u_m|^p\, dx)^{1/p} + \int_{B_4} |f_m|^{q_1}\, dx)^{1/q_1}\right]^p \leq C$$

as a consequence of Lemma (1.4) and (2.4), where $C = C(n, p, q_1, \Lambda)$. Here we have used the assumption that $\delta < 1$. Thus we deduce that

$$I_2 \leq C(\tau)\delta^{(q_2-p)/q_2} + \tau \int_{B_2} |\nabla(u_m - v)|^p dx$$

Estimate of $I_3$. Using Young's inequality with $\tau$ and $Hölder's$ inequality, we have

$$I_3 \leq \tau \int_{B_2} |\nabla(u_m - v)|^p dx + C(\tau) \int_{B_2} |f_m|^p dx$$

$$\leq \tau \int_{B_2} |\nabla(u_m - v)|^p dx + C(\tau)(\int_{B_2} |f_m|^{q_1}\, dx)^{p/q_1}$$

$$\leq \tau \int_{B_2} |\nabla(u_m - v)|^p dx + C(\tau)\delta^p$$

Combining all the estimates of $I_i (1 \leq i \leq 3)$, we obtain

$$C(\tau) \int_{B_2} |\nabla(u_m - v)|^p dx \leq 2\tau \int_{B_2} |\nabla(u_m - v)|^p dx$$

$$+\tau \int_{B_2} |\nabla u_m|^p dx + C(\tau)\left[\delta^{(q_2-p)/q_2} + \delta^p\right]$$

Selecting a small constant $\tau > 0$ such that $0 < \tau \ll \delta < 1$, and then using (2.4), we conclude that

$$\int_{B_2} |\nabla(u - v)|^p dx \leq C\left[\delta + \delta^{(q_2-p)/q_2} + \delta^p\right] = \varepsilon^p$$

by selecting $\delta$ satisfying the last inequality above. This completes the proof.
Let $\delta$ in (1.4) and Definition (1.1) be the same as that in Lemma (2.2). As announced in the beginning of this section, $A$ is $(\delta, 1)$ -vanishing. Therefore

$$\int_{B_i^j} |A - \bar{A}_{B_i^j}| dx \leq \delta \tag{2.7}$$

for $j = 0,1,2,3$, since the radiuses of $B_i^j (0 \le j \le 3)$ are not larger than 1. Then recalling (1.7), we obtain the following scaling invariant form of Lemma (2.2).

**Lemma (2.3):**

Assume that $\lambda \ge \lambda_*$. For any $\varepsilon > 0$, there exists a small $\delta = \delta(\varepsilon) > 0$ such that if $u_m$ is a local weak solution of (1.1) in $\Omega$ with $B_i^3 \subset \Omega$, then there exists $N_0 > 1$ such that

$$\sup_{B_i^2} |\nabla v_\lambda^i| \le N_0 \; and \; \int_{B_i^2} \left|\nabla\left(u_{m\lambda} - v_\lambda^i\right)\right|^p dx \le \varepsilon^p \qquad (2.8)$$

Where $v_\lambda^i$ is the weak solution of (2.1) in $B_i^2$ with $u_{m\lambda}$ replacing $u_m$.

**Proof:** From the definitions of $B_i^j$ for$j = 0,1,2,3$, we rescale by defining

$$\begin{cases} (u_m)_\lambda^i(x) = u_{m\lambda}(2^3\rho_{x_i}x)/(2^3\rho_{z_i}), \\ (f_m)_\lambda^i(x) = f_{m\lambda}(2^3\rho_{x_i}x), \\ A^i(x) = A(2^3\rho_{x_i}x), x \in B_4. \end{cases}$$

Then $(u_m)_\lambda^i$ is a local weak solution of

$$\operatorname{div}((A^i\nabla(u_m)_\lambda^i \cdot \nabla(u_m)_\lambda^i)^{(p-2)/2} A^i \nabla(u_m)_\lambda^i) = \operatorname{div}(|(f_m)_\lambda^i|^{p-2}(f_m)_\lambda^i) \quad \text{in } B_4$$

and from (1.7) and (2.7) one can readily check that

$$\int_{B_4} \left|\nabla(u_m)_\lambda^i(x)\right|^p dx \le 1, \int_{B_4} \left|(f_m)_\lambda^i\right|^p dx \le \delta^p$$

and

$$\int_{B_2} \left|A^i - \overline{A^i}_{B_2}\right|^p dx \le \delta$$

Then according to Lemma (2.1), there exists a weak solution $v$ of

$$\begin{cases} \operatorname{div}((\overline{A^i}_{B_2}\nabla v \cdot \nabla v)^{(p-2)/2}\overline{A^i}_{B_2}\nabla v) = 0 \text{in} B_2, \\ v = u_\lambda^i \text{on } \partial B_2 \end{cases}$$

such that

$$\sup_{B_1} |\nabla v| \le N_0 \text{ and } \int_{B_2} \left|\nabla(u_\lambda^i - v)\right|^p dx \le \varepsilon^p$$

Now we define $v_\lambda^i$ in $B_i^2$ by

$$v(x) = \frac{1}{2^3\rho_{x_i}} v_\lambda^i(2^3\rho_{x_i}x), x \in B_2$$

Then changing variables, we recover the conclusion of Lemma (2.2). This completes the proof.

### 3.3 The main results:

**Theorem (3.1):**

Assume that $q \geq p$. Let $u_m$ be a local weak solution of (1.1). Then there exists a small $\delta = \delta(n, p, q, \Lambda, R) > 0$ so that for each uniformly elliptic and $(\delta, R)$ -vanishing, $A$, and for all $f$ with $f_m \in L^q_{\text{loc}}(\Omega; \mathbb{R}^n)$, we have

$$\int_{B_r(x_0)} |\nabla u_m|^q dx \leq C[\int_{B_{4r}(x_0)} |u_m|^q dx + \int_{B_{4r}(x_0)} |f_m|^q dx] \tag{3.1}$$

where $B_{4r}(x_0) \subset \Omega$ and the constant $C$ is independent of $u_m$ and $f_m$.

Our approach is very much influenced by [2,8].

**Proof:**

i- When $q = p$, the proof is trivial.

ii-From Lemma (2.2), for any $\lambda \geq \lambda_*$ we have

$$\left|\{x \in B_i^1 : |\nabla u_m| > 2N_0\lambda\}\right| = \left|\{x \in B_i^1 : \left|\nabla u_{m\lambda}\right| > 2N_0\}\right|$$

$$\leq \left|\{x \in B_i^1 : \left|\nabla(u_{m\lambda} - v_\lambda^i)\right| > N_0\}\right|$$

$$+ \left|\{x \in B_i^1 : \left|\nabla v_\lambda^i\right| > N_0\}\right| = \left|\{x \in B_i^1 : \left|\nabla(u_{m\lambda} - v_\lambda^i)\right| > N_0\}\right|$$

$$\leq \frac{1}{N_0^p}\int_{B_i^2} \left|\nabla(u_m - v_\lambda^i)\right|^p dz \leq \frac{\varepsilon^p\left|B_i^2\right|}{N_0^p} = \frac{2^{4n}\varepsilon^p\left|B_i^0\right|}{N_0^p}$$

which follows from Lemma (1.5) that

$$\left|\{x \in B_i^1 : |\nabla u_m| > 2N_0\lambda\}\right| \leq C\left(\varepsilon^p(\frac{1}{\lambda^p}\int_{\{x\in B_i^0 : |\nabla u_m| > \lambda/4\}} |\nabla u_m|^p dx\right.$$

$$\left. + \frac{1}{\lambda^{q_1}\delta^{q_1}}\int_{\{x\in B_i^0 : |f| > \delta\lambda/4\}} |f_m|^{q_1} dx\right).$$

where $C = C(n, p, q_1, \Lambda)$. Recalling the fact that the balls $\{B_i^0\}$ are disjoint and

$$\bigcup_{i\in\mathbb{N}} B_i^1 \supset E(\lambda) = \{x \in B_1 : |\nabla u_m| > \lambda\}$$

for any$\lambda \geq \lambda_*$, and then summing up on $i \in \mathbb{N}$ in the inequality above, we have

$$|\{x \in B_1 : |\nabla u_m| > 2N_0\lambda\}|$$

$$\leq \sum_i \left|\{x \in B_i^1 : |\nabla u_m| > 2N_0\lambda\}\right|$$

$$\leq C\varepsilon^p \left( \frac{1}{\lambda^p} \int_{\left\{x\in B_2 : |\nabla u_m| > \frac{\lambda}{4}\right\}} |\nabla u_m|^p dx + \frac{1}{\lambda^{q_1}\delta^{q_1}} \int_{\left\{x\in B_2 : |f| > \frac{\delta\lambda}{4}\right\}} |f_m|^{q_1} dx \right) \qquad (3.2)$$

for any $\lambda \geq \lambda_*$. Recalling the standard argument of measure theory, we compute

$$\int_{B_1} |\nabla u_m|^q dz = q \int_0^\infty \mu^{q-1} |\{x \in B_1 : |\nabla u_m| > \mu\}| d\mu$$

$$= q \int_0^{2N_0\lambda_*} \mu^{q-1} |\{x \in B_1 : |\nabla u_m| > \mu\}| d\mu + q \int_{2N_0\lambda_*}^\infty \mu^{q-1} |\{x \in B_1 : |\nabla u_m| > \mu\}| d\mu$$

$$= q \int_0^{2N_0\lambda_*} \mu^{q-1} |\{x \in B_1 : |\nabla u_m| > \mu\}| d\mu$$

$$+ q \int_{\lambda_*}^\infty (2N_0\lambda)^{q-1} |\{x \in B_1 : |\nabla u_m| > 2N_0\lambda\}| d(2N_0\lambda)$$

$$=: J_1 + J_2.$$

Estimate of $J_1$: From the definitions of $\lambda_*$ and $\lambda_0$ we deduce that

$$\lambda_*^q = 2^{6nq/p}\lambda_0^q \leq C\{(\int_{B_2} |\nabla u_m|^p dx)^{q/p} + \frac{1}{\delta^q}(\int_{B_2} |f_m|^{q_1} dx)^{q/q_1}\}$$

which follows from Lemma (1.3) and *Hölder's* inequality that

$$\lambda_*^q \leq C\left[\left(\int_{B_4} |u_m|^p dx + \int_{B_4} |f_m|^p dx\right)^{q/p} + \frac{1}{\delta^q}\left(\int_{B_2} |f_m|^{q_1} dx\right)^{q/q_1}\right]$$

$$\leq C\{(\int_{B_4} |u_m|^p dx)^{q/p} + (\int_{B_4} |f_m|^p dx)^{q/p} + \frac{1}{\delta^q}\int_{B_2} |f_m|^q dx\}$$

$$\leq C\{\int_{B_4} |u_m|^q dx + \int_{B_4} |f_m|^q dx\}$$

Therefore, we discover

$$J_1 \leq (2N_0\lambda_*)^q |B_1| \leq C\left\{\int_{B_4} |u_m|^q dx + \int_{B_4} |f_m|^q dx\right\}$$

where $C = C(n, p, q, \Lambda)$.
Estimate of $J_2$. From (3.2) we deduce that

$$J_2 \leq C\varepsilon^p \left\{\int_0^\infty \lambda^{q-p-1} \int_{\{x\in B_2:|\nabla u_m|>\lambda/4\}} |\nabla u_m|^p dx d\lambda\right.$$

$$\left. + \frac{1}{\delta^{q_1}} \int_0^\infty \lambda^{q-q_1-1} \int_{\{x\in B_2:|f_m|>\delta\lambda/4\}} |f_m|^{q_1} dx d\lambda\right\}$$

Recalling that

$$\int_{\mathbb{R}^n} |g|^\beta dx = (\beta - \alpha) \int_0^\infty \mu^{\beta-\alpha-1} \int_{\{x\in\mathbb{R}^n:|g|>\mu\}} g^\alpha dx d\mu$$

for $\beta > \alpha > 1$, we have

$$J_2 \leq C_1\varepsilon^p \int_{B_2} |\nabla u_m|^q dx + C_2\varepsilon^p \int_{B_2} |f_m|^q dx$$

where $C_1 = C_1(n, p, q, \Lambda)$ and $C_2 = C_2(n, p, q, \Lambda, \delta)$ .
Combining the estimates of $J_1$ and $J_2$ we obtain

$$\int_{B_1} |\nabla u_m|^q dx \leq C_1\varepsilon^p \int_{B_2} |\nabla u_m|^q dx + C_3 \int_{B_4} (|u_m|^q + |f_m|^q) dx$$

where $C_3 = C_3(n, p, q, \Lambda, \delta, \varepsilon)$. Selecting suitable $\varepsilon$ such that $C_1\varepsilon^p = 1/2$, and reabsorbing at the right-hand side first integral in the inequality above by a covering and iteration argument (see [5], [7]), we have

$$\int_{B_1} |\nabla u_m|^q dx \leq C\{\int_{B_4} |u_m|^q dx + \int_{B_4} |f_m|^q dx$$

Then by a shift and scaling transform, we can finish the proof of the main result.

**Corollary (3.2):**

Assume that $\varepsilon > 0$, let $u_m$ be a sequence of local weak solutions of (1.1). Then there exists a small $\delta = \delta(n, 1+\varepsilon, \frac{1+\varepsilon}{\varepsilon}, \Lambda, R) > 0$ so that for each uniformly elliptic and $(\delta, R)$ -vanishing,$A$, and for all $f_m$ with $f_m \in L_{\text{loc}}^{\frac{1+\varepsilon}{\varepsilon}}(\Omega; \mathbb{R}^n)$, we have

$$\int_{B_r(x_0)} \sum_{m=1}^{s} |\nabla u_m|^{\frac{1+\varepsilon}{\varepsilon}} dx$$

$$\leq \tilde{C}[\int_{B_{4r}(x_0)} \sum_{m=1}^{s} |u_m|^{\frac{1+\varepsilon}{\varepsilon}} dx + \int_{B_{4r}(x_0)} \sum_{m=1}^{s} |f_m|^{\frac{1+\varepsilon}{\varepsilon}} dx]$$

(120) where $B_{4r}(x_0) \subset \Omega$ and the constant $\tilde{C}$ is independent of $u_m$ and $f_m$.

**Proof:** From Lemma (2.2), for any $\lambda = \lambda_* + \varepsilon$ we have

$$1/\left(2x \in B_i^1 : \sum_{m=1}^{s} |\nabla u_m| > 2N_0(\lambda_* + \varepsilon)\right)^{1/2} = 1/\left\{x \in B_i^1 : \sum_{m=1}^{s} \left|\nabla (u_m)_{\lambda_*+\varepsilon}\right| > 2N_0\right\}^{1/2}$$

$$\leq 1/\{x \in B_i^1 : \sum_{m=1}^{s} \left|\nabla((u_m)_{\lambda_*+\varepsilon} - v^i_{\lambda_*+\varepsilon})\right| > N_0\}^{1/2} + \left|\{x \in B_i^1 : \left|\nabla v^i_{\lambda_*+\varepsilon}\right| > N_0\}\right|$$

$$= 1/\{x \in B_i^1 : \textstyle\sum_{m=1}^{s} \left|\nabla((u_m)_{\lambda_*+\varepsilon} - v^i_{\lambda_*+\varepsilon})\right| > N_0\}^{1/2}$$

$$\leq \frac{1}{N_0^{1+\varepsilon}} \int_{B_i^2} \sum_{m=1}^{s} \left|\nabla((u_m)_{\lambda_*+\varepsilon} - v^i_{\lambda_*+\varepsilon})\right|^{1+\varepsilon} dz \leq \frac{\varepsilon^{1+\varepsilon}\left|B_i^2\right|}{N_0^{1+\varepsilon}} = \frac{2^{4n}\varepsilon^{1+\varepsilon}\left|B_i^0\right|}{N_0^{1+\varepsilon}},$$

which follows from Lemma (1.5) that

$$1/\{2x \in B_i^1 : \sum_{m=1}^{s} |\nabla u_m| > 2N_0(\lambda_* + \varepsilon)\}^{1/2}$$

$$\leq \tilde{C}\varepsilon^{1+\varepsilon}(\frac{1}{\lambda^{1+\varepsilon}} \int_{\{x \in B_i^0 : \sum_{m=1}^{s} |\nabla u_m| > (\lambda_*+\varepsilon)/4\}} \sum_{m=1}^{s} |\nabla u_m|^{1+\varepsilon} dx$$

$$+ \frac{1}{\lambda^{(\frac{1+\varepsilon}{\varepsilon})_1} \delta^{(\frac{1+\varepsilon}{\varepsilon})_1}} \int_{\{x \in B_i^0 : \sum_{m=1}^{s} |f_m| > \delta(\lambda_*+\varepsilon)/4\}} \sum_{m=1}^{s} |f_m|^{(\frac{1+\varepsilon}{\varepsilon})_1} dx)$$

where $\tilde{C} = \tilde{C}(n, 1+\varepsilon, (\frac{1+\varepsilon}{\varepsilon})_1, \Lambda)$. Recalling the fact that the balls $\{B_i^0\}$ are disjoint and

$$\bigcup_{i\in\mathbb{N}} B_i^1 \supset E((\lambda_* + \varepsilon)) = \{x \in B_1 : \sum_{m=1}^{s} |\nabla u_m| > (\lambda_* + \varepsilon)\}$$

for any $\lambda = \lambda_* + \varepsilon$, and then summing up on $i \in \square$ in the inequality above, we have

$$1/\{2x \in B_1 : \sum_{m=1}^{s} |\nabla u_m| > 2N_0(\lambda_* + \varepsilon)\}^{1/2}$$

$$\leq \sum_i 1/\{2x \in B_i^1 : \sum_{m=1}^{s} |\nabla u_m| > 2N_0(\lambda_* + \varepsilon)\}^{1/2}$$

$$\leq \tilde{C}\varepsilon^{1+\varepsilon}\Big(\frac{1}{\lambda^{1+\varepsilon}} \int_{\{x\in B_2 : \sum_{m=1}^{s} |\nabla u_m| > (\lambda_* + \varepsilon)/4\}} \sum_{m=1}^{s} |\nabla u_m|^{1+\varepsilon}\, dx$$

$$\frac{1}{\lambda^{(\frac{1+\varepsilon}{\varepsilon})_1} \delta^{(\frac{1+\varepsilon}{\varepsilon})_1}} \int_{\{x\in B_2 : \sum_{m=1}^{s} |f_m| > \delta(\lambda_* + \varepsilon)/4\}} \sum_{m=1}^{s} |f_m|^{(\frac{1+\varepsilon}{\varepsilon})_1}\, dx\Big) \qquad (3.3)$$

for any $\lambda = \lambda_* + \varepsilon$. Recalling the standard argument of measure theory, we compute

$$\int_{B_1} \sum_{m=1}^{s} |\nabla u_m|^{(\frac{1+\varepsilon}{\varepsilon})}\, dz$$

$$= (\frac{1+\varepsilon}{\varepsilon}) \int_0^{\infty} \mu^{(\frac{1+\varepsilon}{\varepsilon})-1} 1/\{2x \in B_1 : \sum_{m=1}^{s} |\nabla u_m| > \mu\}^{1/2} d\mu$$

$$= (\frac{1+\varepsilon}{\varepsilon}) \int_0^{2N_0\lambda_*} \mu^{(\frac{1+\varepsilon}{\varepsilon})-1} 1/\{2x \in B_1 : \sum_{m=1}^{s} |\nabla u_m| > \mu\}^{1/2} d\mu$$

$$+(\frac{1+\varepsilon}{\varepsilon}) \int_{2N_0\lambda_*}^{\infty} \mu^{(\frac{1+\varepsilon}{\varepsilon})-1} 1/\{2x \in B_1 : \sum_{m=1}^{s} |\nabla u_m| > \mu\}^{1/2} d\mu$$

$$= (\frac{1+\varepsilon}{\varepsilon}) \int_0^{2N_0\lambda_*} \mu^{(\frac{1+\varepsilon}{\varepsilon})-1} 1/\{2x \in B_1 : \sum_{m=1}^{s} |\nabla u_m| > \mu\}^{1/2} d\mu$$

$$+(\frac{1+\varepsilon}{\varepsilon}) \int_{\lambda_*}^{\infty} (2N_0(\lambda_* + \varepsilon))^{(\frac{1+\varepsilon}{\varepsilon})-1} K.d(2N_0(\lambda_* + \varepsilon)\}$$

where,

$$K = \left(1/2x \in B_1 : \sum_{m=1}^{s} |\nabla u_m| > 2N_0(\lambda_* + \varepsilon)\right)^{1/2}$$

$$=: J_1 + J_2.$$

Estimate of $J_1$. From the definitions of $\lambda_*$ and $\lambda_0$ we deduce that

$$\lambda_*^{(\frac{1+\varepsilon}{\varepsilon})} = 2^{6n(\frac{1+\varepsilon}{\varepsilon})/(1+\varepsilon)} \lambda_0^{(\frac{1+\varepsilon}{\varepsilon})} \leq \tilde{C}\{(\int_{B_2} \sum_{m=1}^{s} |\nabla u_m|^{(1+\varepsilon)}\, dx)^{(\frac{1+\varepsilon}{\varepsilon})/(1+\varepsilon)}$$

$$+ \frac{1}{\delta^{(\frac{1+\varepsilon}{\varepsilon})}} (\int_{B_2} \sum_{m=1}^{s} |f_m|^{(\frac{1+\varepsilon}{\varepsilon})_1}\, dx)^{(\frac{1+\varepsilon}{\varepsilon})/(\frac{1+\varepsilon}{\varepsilon})_1}\}$$

which follows from Lemma (1.3) and $H\ddot{o}lder's$ inequality that

$$\lambda_*^{(\frac{1+\varepsilon}{\varepsilon})} \leq \tilde{C}\{(\int_{B_4} \sum_{m=1}^{s} |u_m|^{(1+\varepsilon)}\, dx + \int_{B_4} \sum_{m=1}^{s} |f_m|^{(1+\varepsilon)}\, dx)^{(\frac{1+\varepsilon}{\varepsilon})/(1+\varepsilon)}$$

$$+ \frac{1}{\delta^{(\frac{1+\varepsilon}{\varepsilon})}} (\int_{B_2} \sum_{m=1}^{s} |f_m|^{(\frac{1+\varepsilon}{\varepsilon})_1}\, dx)^{(\frac{1+\varepsilon}{\varepsilon})/(\frac{1+\varepsilon}{\varepsilon})_1}\}$$

$$\leq \tilde{C}\{(\int_{B_4} \sum_{m=1}^{s} |u_m|^{(1+\varepsilon)}\, dx)^{(\frac{1+\varepsilon}{\varepsilon})/(1+\varepsilon)} + (\int_{B_4} \sum_{m=1}^{s} |f_m|^{(1+\varepsilon)}\, dx)^{(\frac{1+\varepsilon}{\varepsilon})/(1+\varepsilon)}$$

$$+ \frac{1}{\delta^{(\frac{1+\varepsilon}{\varepsilon})}} \int_{B_2} \sum_{m=1}^{s} |f_m|^{(\frac{1+\varepsilon}{\varepsilon})}\, dx\}$$

$$\leq \tilde{C}\{\int_{B_4} \sum_{m=1}^{s} |u_m|^{(\frac{1+\varepsilon}{\varepsilon})}\, dx + \int_{B_4} \sum_{m=1}^{s} |f_m|^{(\frac{1+\varepsilon}{\varepsilon})}\, dx\}$$

Therefore, we discover

$$J_1 \leq (2N_0\lambda_*)^{(\frac{1+\varepsilon}{\varepsilon})} |B_1| \leq \tilde{C}\{\int_{B_4} \sum_{m=1}^{s} |u_m|^{(\frac{1+\varepsilon}{\varepsilon})}\, dx + \int_{B_4} \sum_{m=1}^{s} |f_m|^{(\frac{1+\varepsilon}{\varepsilon})}\, dx\}$$

where $\tilde{C} = \tilde{C}(n, 1+\varepsilon, (\frac{1+\varepsilon}{\varepsilon}), \Lambda)$.

Estimate of $J_2$. From (3.3) we deduce that

$$J_2 \leq$$

$$\tilde{C}\varepsilon^{(1+\varepsilon)}\{\int_0^\infty (\lambda_*+\varepsilon)^{(\frac{1+\varepsilon}{\varepsilon})-(1+\varepsilon)-1}\int_{\{x\in B_2:\sum_{m=1}^s|\nabla u_m|>(\lambda_*+\varepsilon)/4\}}\sum_{m=1}^s|\nabla u_m|^{(1+\varepsilon)}\,dxd(\lambda_*+\varepsilon)$$

$$+\frac{1}{\delta^{(\frac{1+\varepsilon}{\varepsilon})_1}}\int_0^\infty (\lambda_*+\varepsilon)^{(\frac{1+\varepsilon}{\varepsilon})-(\frac{1+\varepsilon}{\varepsilon})_1-1}\int_{\{x\in B_2:\sum_{m=1}^s|f_m|>\delta(\lambda_*+\varepsilon)/4\}}\sum_{m=1}^s|f_m|^{(\frac{1+\varepsilon}{\varepsilon})_1}\,dxd(\lambda_*+\varepsilon)\}$$

Recalling that

$$\int_{\mathbb{R}^n}|g|^\beta dx=(\beta-\alpha)\int_0^\infty \mu^{\beta-\alpha-1}\int_{\{x\in\mathbb{R}^n:|g|>\mu\}}g^\alpha dxd\mu$$

for $\beta>\alpha>1$, we have

$$J_2\le\tilde{C}_1\varepsilon^{(1+\varepsilon)}\int_{B_2}\sum_{m=1}^s|\nabla u_m|^{(\frac{1+\varepsilon}{\varepsilon})}\,dx+\tilde{C}_2\varepsilon^{(1+\varepsilon)}\int_{B_2}\sum_{m=1}^s|f_m|^{(\frac{1+\varepsilon}{\varepsilon})}\,dx$$

where $\tilde{C}_1=\tilde{C}_1(n,1+\varepsilon,(\frac{1+\varepsilon}{\varepsilon}),\Lambda)$ and $\tilde{C}_2=\tilde{C}_2(n,1+\varepsilon,(\frac{1+\varepsilon}{\varepsilon}),\Lambda)$ .
Combining the estimates of $J_1$ and $J_2$ we obtain

$$\int_{B_1}\sum_{m=1}^s|\nabla u_m|^{(\frac{1+\varepsilon}{\varepsilon})}\,dx\le\tilde{C}_1\varepsilon^{(1+\varepsilon)}\int_{B_2}\sum_{m=1}^s|\nabla u_m|^{(\frac{1+\varepsilon}{\varepsilon})}\,dx$$
$$+\tilde{C}_3\int_{B_4}\sum_{m=1}^s(|u_m|^{(\frac{1+\varepsilon}{\varepsilon})}+|f_m|^{(\frac{1+\varepsilon}{\varepsilon})})\,dx$$

where $\tilde{C}_3=\tilde{C}_3(n,1+\varepsilon,(\frac{1+\varepsilon}{\varepsilon}),\Lambda,\delta,\varepsilon)$. Selecting suitable $\varepsilon$ such that $\tilde{C}_1\varepsilon^{(1+\varepsilon)}=1/2$, and reabsorbing at the right-hand side first integral in the inequality above by a covering and iteration argument, we have

$$\int_{B_1}\sum_{m=1}^s|\nabla u_m|^{(\frac{1+\varepsilon}{\varepsilon})}\,dx\le\tilde{C}\{\int_{B_4}\sum_{m=1}^s|u_m|^{(\frac{1+\varepsilon}{\varepsilon})}\,dx+\int_{B_4}\sum_{m=1}^s|f_m|^{(\frac{1+\varepsilon}{\varepsilon})}\,dx$$

Then by a shift and scaling transform, we can finish the proof.

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